2 Yield Criteria

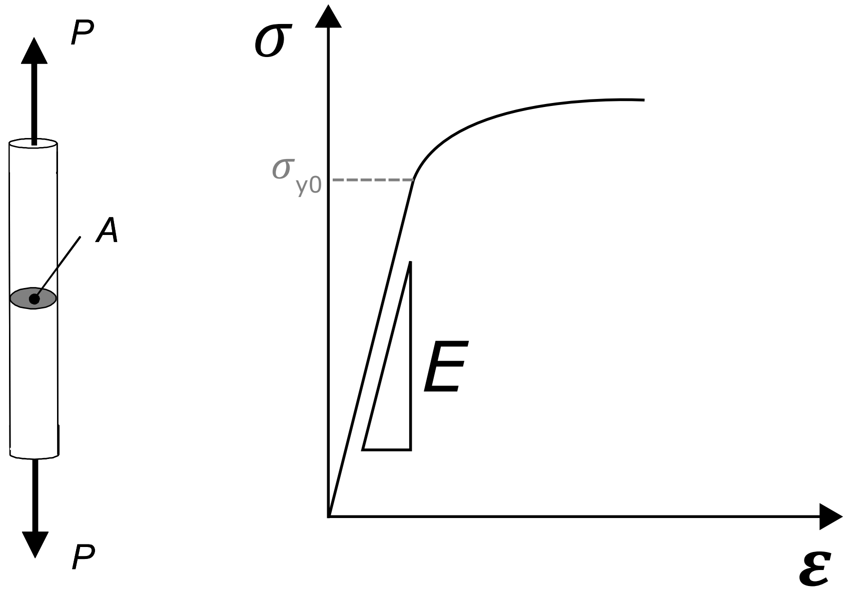
Learning Summary

1. Recognise the difference between ductile and brittle failure, as illustrated by the behaviour of bars subjected to uniaxial tension and torsion (knowledge);
2. Describe the meaning of yield stress and proof stress, in uniaxial tension, for a material (comprehension);
3. Describe the Tresca (maximum shear stress) yield criterion and the 2D and 3D diagrammatic representations of it (comprehension);
4. Employ the Tresca yield criterion to determine whether yield has occurred in a structure (application);
5. Describe the von Mises (maximum shear strain energy) yield criterion and the 2D and 3D diagrammatic representations of it (comprehension);
6. Employ the von Mises yield criterion to determine whether yield has occurred in a structure (application).

2.1 Introduction

If a ductile material is subjected to uniaxial loading, as shown schematically in Figure 2.1(a), beyond a certain point (the initial yield stress, σ_{y0}) the stress-strain behaviour ceases to be linear (i.e. stress is no longer proportional to strain) as shown in Figure 2.1(b) and the material is said to have 'yielded'. Typical stress-strain curves for some materials are shown schematically in Figure 2.2. For some materials (such as the aluminium alloy in Figure 2.2), the yield stress is not easily discernible, in these cases an offset or proof stress is defined, commonly at 0.1 or 0.2% strain. This value is determined by drawing a line at the same gradient as the elastic portion of the stress-strain curve starting at at the value of strain at which the proof stress is to be determined, as shown in Figure 2.3. Loading past the yield stress, σ_{y0}, leads to permanent, unrecoverable deformation (permanent strain) of the material when the stress is removed, as shown by the quantity x in Figure 2.4.

Materials that can be subjected to large strains before failure, such as mild steel and the aluminium alloy in Figure 2.2, are known as ductile materials, these are good materials from an engineering perspective as they absorb a lot of energy and exhibit large deformations before failure. In contrast, grey cast iron is a brittle material and shows little or no yielding before failure.



(a) (b)

Figure 2.1

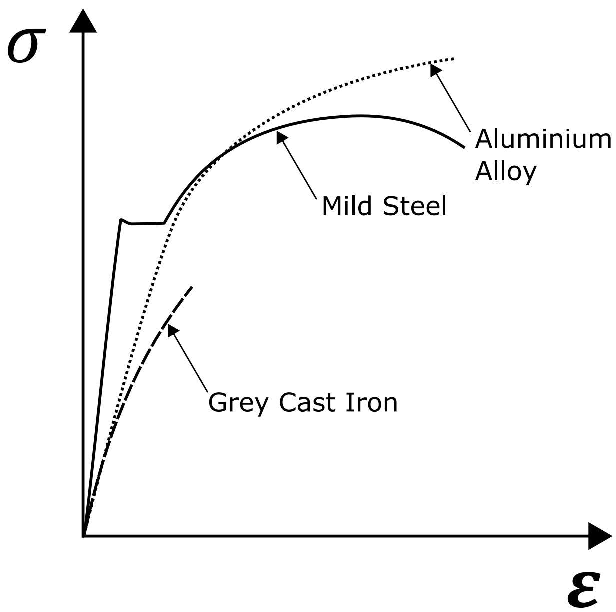


Figure 2.2

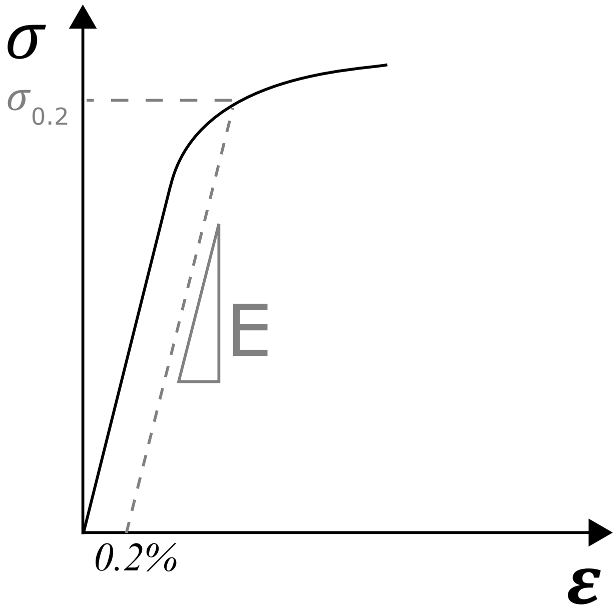


Figure 2.3

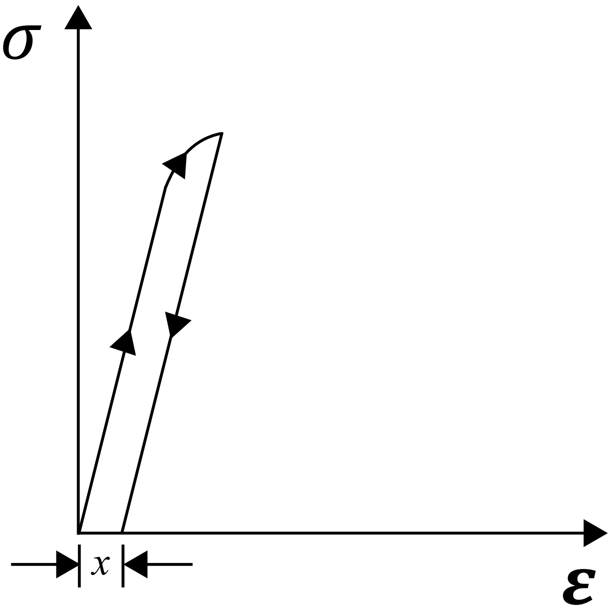


Figure 2.4

The ductility of a material is usually expressed as a percentage elongation or percentage reduction in area at failure. The percentage elongation is the failure strain of the sample expressed as a percent. If L_f is the final length of the specimen at failure and L_0 is the original length, the percent elengation is expressed as:

$$Percent\ elongation = \frac{L_f - L_0}{L_0}(100\%) \tag{2.1}$$

The percentage reduction in area can be expressed as:

$$Percent\ reduction\ in\ area = \frac{A_0 - A_f}{A_0}(100\%) \qquad (2.2)$$

where A_0 and A_f are the initial and final cross sectional areas respectively.

2.2 Failure of Ductile Materials

The failure of a tensile test specimen of a ductile material, such as mild steel, tends to be a 'cup and cone' mode of failure as shown in Figure 2.5(a) with a cone angle of 45°. Analysis of the Mohr's circle for the loading condition, as shown in Figure 2.5(b) indicates that the 45° plane is the plane on which the maximum shear stress occurs.

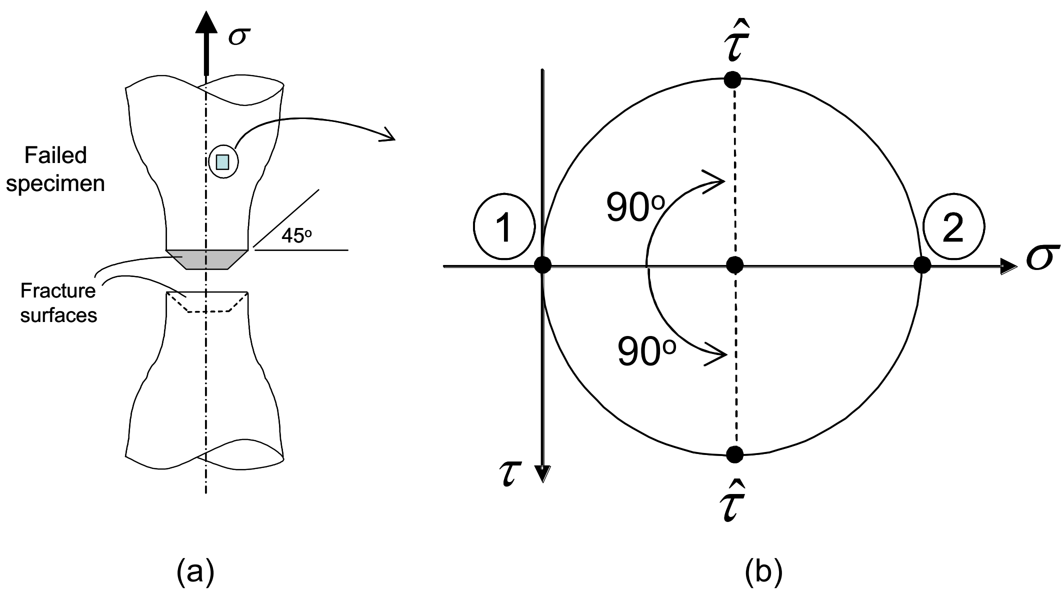


Figure 2.5

If a circular cross section bar of ductile material (mild steel in this case) is subjected to torsional loading as shown in Figure 2.6(a), the torque-angle response is shown in Figure 2.6(b). For T ≤ T_y, $T \propto \theta$ and the torque-angle behaviour is reversed on removal of the torque. If the torque is continuously increased until failure occurs, the fracture plane is transverse to the axis of the specimen as shown in Figure 2.7. A Mohr's circle for the torsion loading case, also shown, indicates that failure occurs on the maximum shear stress plane.

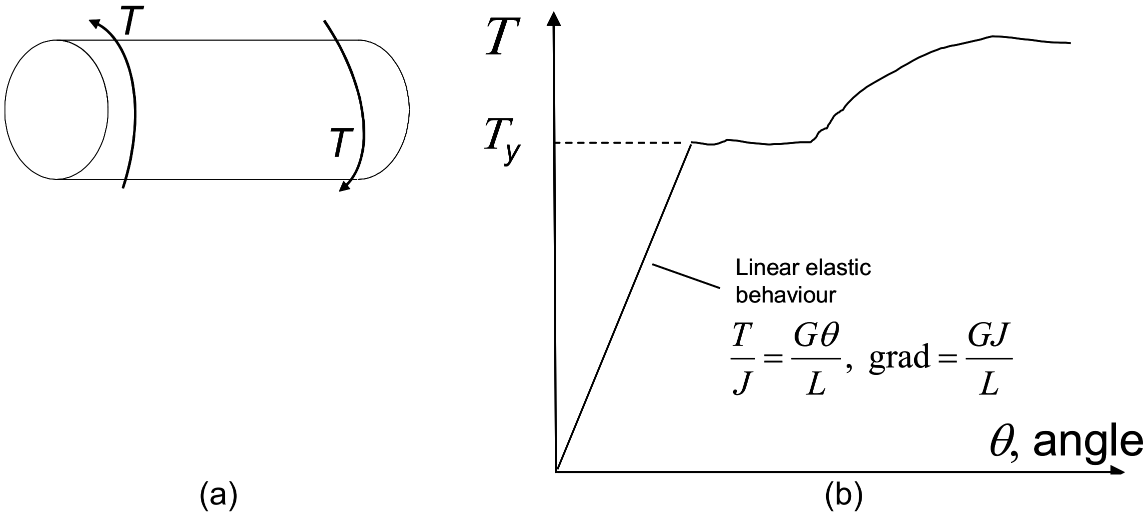


Figure 2.6

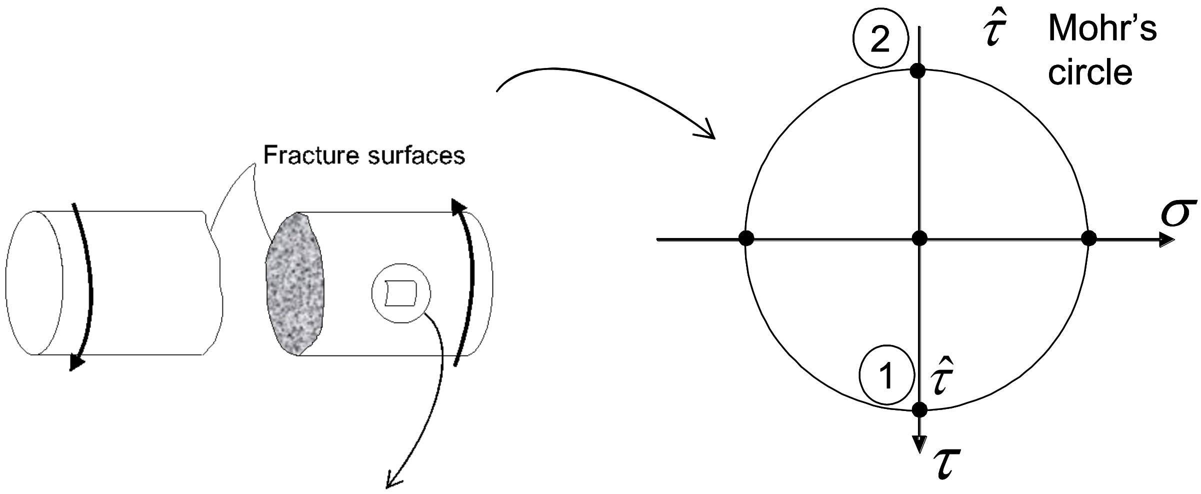


Figure 2.7

Therefore, the results of both the tension and the torsion tests indicate that failure occurs on the planes that contain the maximum shear stresses. This behaviour is similar to that of many "ductile" materials.

2.3 Failure of Brittle Materials

The failure of a tensile test of a brittle material, such as grey cast iron, will tend to be a flat fracture surface perpendicular to the loading direction, as shown in Figure 2.8. As previously mentioned there is little or no plastic deformation and the crack propagation across the fracture surface is very fast.

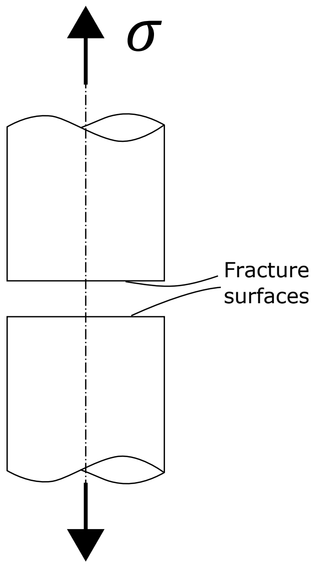


Figure 2.8

A torsion test on a brittle material will lead to a torque-angle response as shown in Figure 2.9 and will lead to a 45° helical failure in the specimen as shown in Figure 2.10.

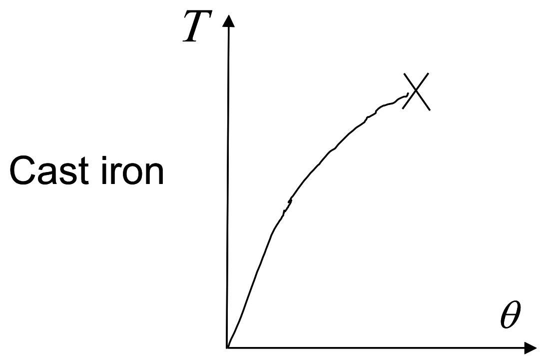


Figure 2.9

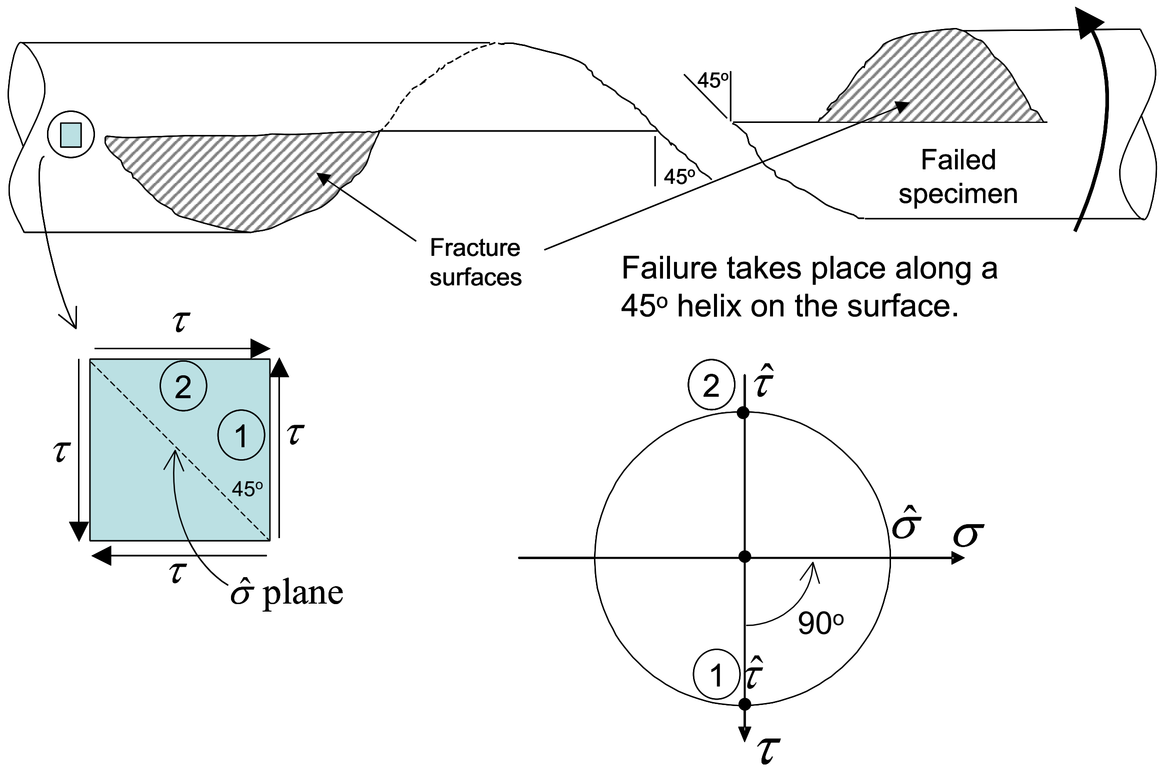


Figure 2.10

A Mohr's circle for this loading condition, also shown in Figure 2.10, shows that the point on the Mohr's circle associated with the maximum principal stress is 90° (ccw) from point 1. Therefore, this represents a plane at 45° from plane 1 on the element. This maximum principal stress, $\hat{\sigma}$, plane corresponds to the 45° helix angle of the fracture on the surface. This indicates that failure in this material has occurred on a plane on which the maximum principal stress exists.

The tests and stress states used to come to the above conclusions for "ductile" and "brittle" failures are very simple and it would therefore be unwise to base failure criteria on this evidence alone.

2.4 Yielding of Ductile Materials

The topic of "Yield Criteria" is limited to the prediction of the initiation of yielding in "ductile" materials.

Two criteria that generally provide a good indication of yield that are widely used in elastic-plastic analysis are the maximum shear stress (Tresca) criterion and the maximum shear strain energy (von Mises) criterion.

2.5 The Maximum Shear Stress (Tresca) Yield Criterion

If σ_1, σ_2 and σ_3 are the principal stresses in three-dimensions ($\sigma_1 > \sigma_2 > \sigma_2$) then as shown in the Mohr's circle in Figure 2.11:

$$\tau_{max} = \frac{\sigma_3 - \sigma_1}{2} \tag{2.3}$$

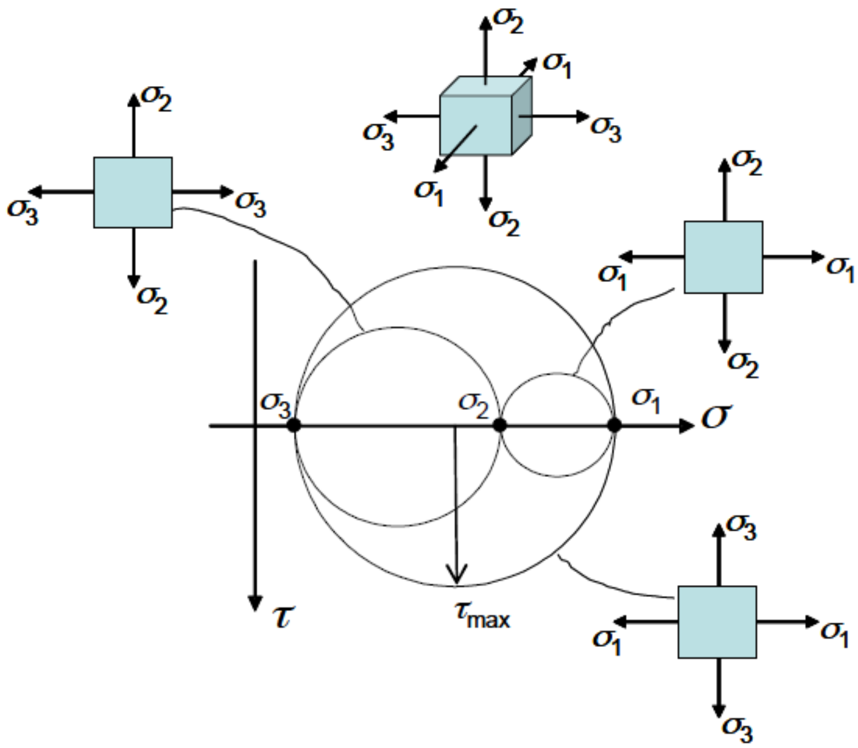


Figure 2.11

The Tresca yield criterion states that the material will yield when the maximum shear stress in the material exceeds a limiting value, this limiting value can be related to the uniaxial yield stress, σ_y when $\sigma_1 = \sigma_y$, σ_2 σ_3 = 0

$$\tau_{max} = \frac{\sigma_y - 0}{2} = \frac{\sigma_y}{2} \tag{2.4}$$

The Tresca (or τ_{max}) yield criterion therefore states that the material will yield if

$$\sigma_1 - \sigma_3 \geq \sigma_y \text{ for } \sigma_1 > \sigma_2 > \sigma_3 \quad (2.5)$$

2.6 The Maximum Shear Strain Energy (von Mises) Yield Criterion

The von Mises yield criterion states that the material will yield when the maximum shear strain energy (per unit volume) exceeds a limiting value. If σ_1, σ_2 and σ_3 are the three principal stresses ($\sigma_1 > \sigma_2 > \sigma_3$) then:

$$\frac{\text{shear strain energy}}{\text{unit volume}} = \frac{1}{12G}\{(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2\} \quad (2.6)$$

Again, the limiting value can be related to the uniaxial yield stress, σ_y, obtained from a uniaxial tensile test. Thus, at yield when $\sigma_1 = \sigma_y$, σ_2 σ_3 = 0:

$$\frac{\text{shear strain energy}}{\text{unit volume}} = \frac{1}{12G}\{2{\sigma_y}^2\} \quad (2.7)$$

The von Mises yield criterion can thus be expressed as follows:

$$\frac{1}{12G}\{(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2\} \geq \frac{1}{12G}\{2{\sigma_y}^2\} \quad (2.8)$$

which can be reduced to the following, more common expression for the onset of yield, according to the von Mises yield criterion:

$$(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2 \geq 2{\sigma_y}^2 \quad (2.9)$$

2.7 Two-dimensional Stress Systems (i.e. σ_3 = 0)

The yield boundaries for both the Tresca and von Mises in a two-dimensional stress-state are shown in Figure 2.12. For plotting purposes here, σ_1 and σ_2 can take on any values, i.e. σ_1 is not necessarily always greater than σ_2.

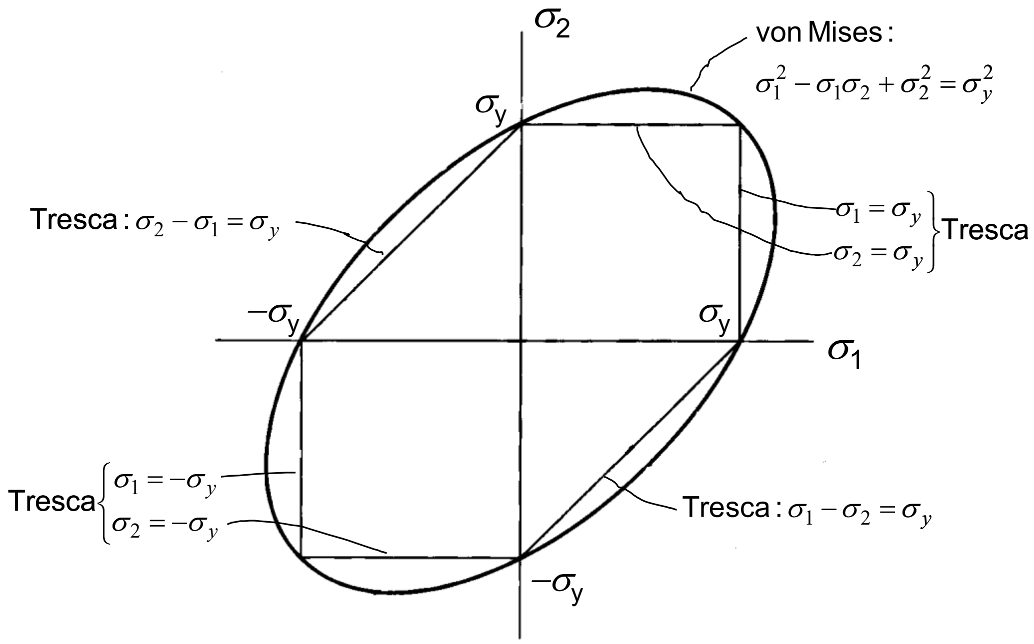


Figure 2.12

In general, the von Mises yield criterion is easier to handle analytically because it is continuous. This is particularly important for the calculation of incremental plastic strains, since the plastic strains are related to the normal to the yield surface and at the corners of the Tresca yield locus there is ambiguity about the directions of the normal, whereas there is no such ambiguity about the von Mises yield locus.

2.8 Three-dimensional Stress Systems

The Tresca and von Mises yield criterion are not altered if a constant stress component (σ) is added to each stress component:

$$(\sigma_1 + \sigma) - (\sigma_3 + \sigma) = \sigma_1 - \sigma_3 = \sigma_y \tag{2.10}$$

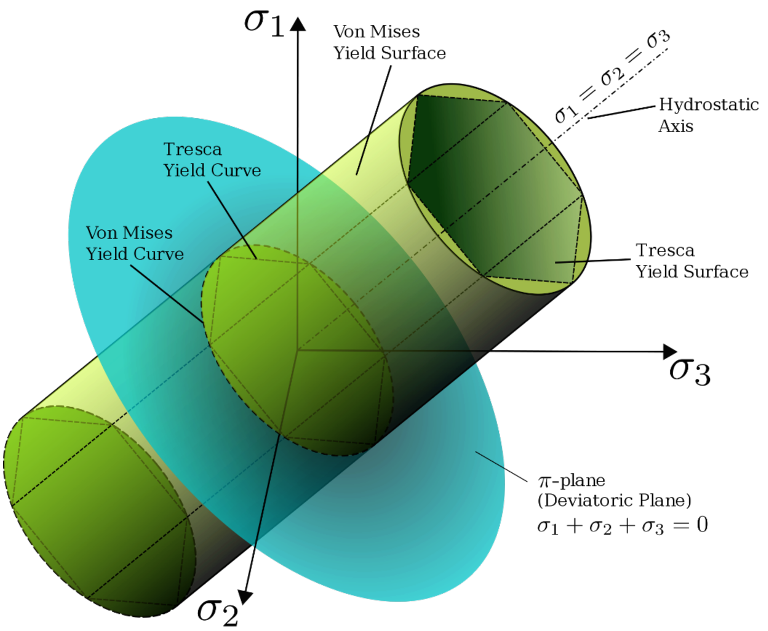
and

$$
\begin{aligned}
&\big((\sigma_1+\sigma)-(\sigma_2+\sigma)\big)^2+\big((\sigma_2+\sigma)-(\sigma_3+\sigma)\big)^2 \\
&\qquad +\big((\sigma_3+\sigma)-(\sigma_1+\sigma)\big)^2 \\
&\qquad =(\sigma_1-\sigma_2)^2+\ (\sigma_2-\sigma_3)^2+(\sigma_3-\sigma_1)^2=2{\sigma_y}^2
\end{aligned}
\tag{2.11}
$$

This implies that the addition of a "hydrostatic stress state", i.e. $\sigma_1 = \sigma_2 = \sigma_3 = \sigma$ does not change the shapes of the yield surfaces shown in the section on two-dimensional stress systems.

The mean principal stress $\sigma_h = \frac{1}{3}(\sigma_1+\sigma_2+\sigma_3)$, which is known as the *hydrostatic stress* for a given stress state ($\sigma_1, \sigma_2, \sigma_3$), is the stress which causes volume change. Now, the independence of the yield criteria with respect to hydrostatic stress means that the three-dimensional yield criteria are prismatic surfaces with the axes of the prisms in each case being the line $\sigma_1 = \sigma_2 = \sigma_3$. This is called the *hydrostatic line* in 3D stress space (Haigh-Westergaard stress space) and it has direction cosines $(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}})$. The yield boundaries can thus move any distance in the direction $\sigma_1 = \sigma_2 = \sigma_3$. The yield surfaces for both the von Mises and Tresca yield criteria therefore have a constant oblique section and hence a constant perpendicular cross-section, whose true shape can be seen in the view along the line $\sigma_1 = \sigma_2 = \sigma_3$. Any arbitrary stress 'vector' ($\sigma_1, \sigma_2, \sigma_3$), e.g. $\overrightarrow{OB}$ and$\overrightarrow{OD}$, (Figure 2.13), in the stress space can be decomposed into two components, one parallel to the hydrostatic line, e.g. $\overrightarrow{OA}$ and $\overrightarrow{OC}$, and one perpendicular to the hydrostatic line, e.g. $\overrightarrow{AB}$ and $\overrightarrow{CD}$. The oblique planes which are perpendicular to the hydrostatic line are called deviatoric planes and are given by equations of the form $\sigma_1 + \sigma_2 + \sigma_3 = const$, each representing a different level of hydrostatic stress. The deviatoric plane with $\sigma_1 + \sigma_2 + \sigma_3 = 0$ is known as the π-plane. It can be shown that the component of ($\sigma_1, \sigma_2, \sigma_3$) parallel to the hydrostatic line is ($\sigma_h, \sigma_h, \sigma_h$), e.g. $\overrightarrow{OA}$ and $\overrightarrow{OC}$, while the component parallel to the deviatoric planes is $(\sigma_1-\sigma_h, \sigma_2-\sigma_h, \sigma_3-\sigma_h)$, $\overrightarrow{AB}$ and $\overrightarrow{CD}$. Only the latter component of stress is important in determining yield according to the von Mises and Tresca criteria.

The representation of yield surfaces for a three-dimensional stress state and the decomposition of the stress into hydrostatic and deviatoric components are shown in Figure 2.13.



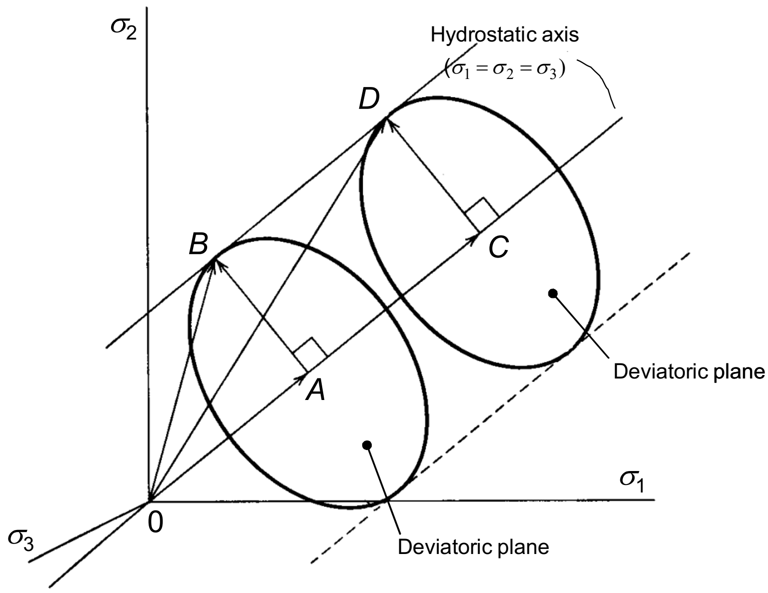


Figure 2.13

The view along the $\sigma_1 = \sigma_2 = \sigma_3$ line of the von Mises and Tresca yield criteria is an isometric view showing the three axes included at 120° intervals. This is sometimes called a view on the π-plane, as shown in Figure 2.14 on which the Tresca yield surface is a hexagon and the von Mises yield surface is a circle. Therefore, large principal stresses do not necessarily result in yield; it is the stress differences and the route to the final stress state that govern whether yielding will occur.

Figure 2.14 can be used, instead of the equations, to decide whether a certain stress state will be safe. Simply plot on the diagram each of the three principal stresses parallel to each of the three axes and see whether the final point lies inside the appropriate yield surface.

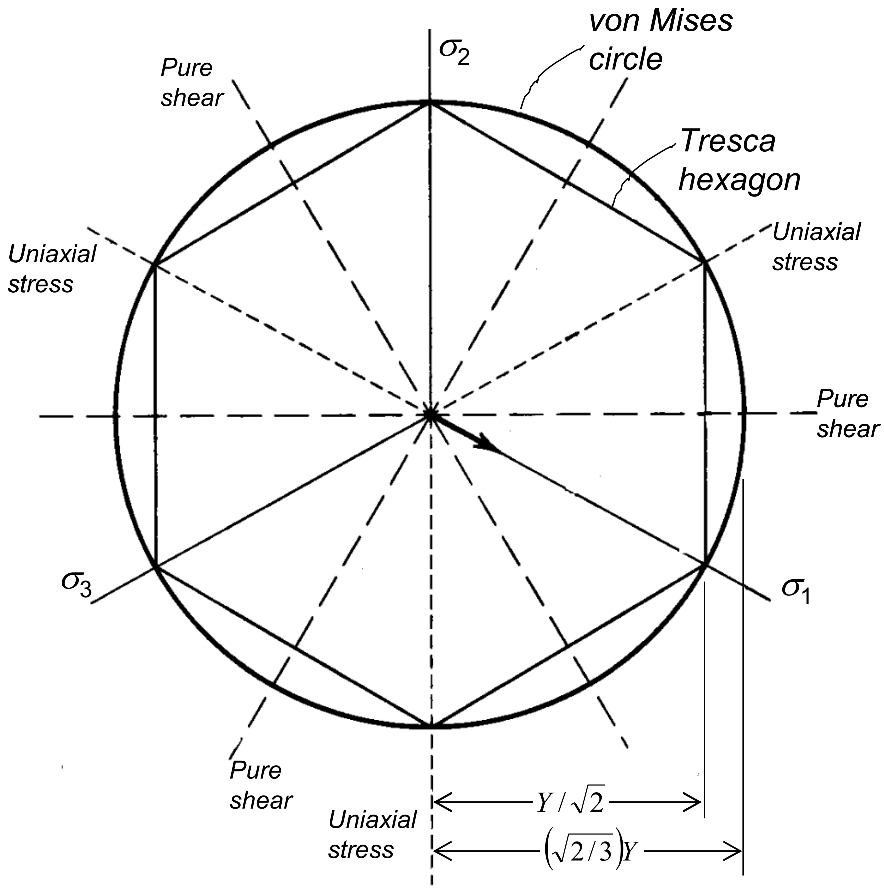


Figure 2.14

NB:

(i) The yield condition can be examined by either using the appropriate equation, i.e.

Tresca: $\sigma_1 - \sigma_3 = \sigma_y \;\; (\sigma_1 > \sigma_2 > \sigma_3)$

von Mises: $(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2 = 2{\sigma_y}^2$

or by plotting principal stresses on the π-plane.

(ii) All three principal stresses are important. At free surfaces the normal stress is usually zero, but it may be important, particularly if the other two principal stresses are of the same sign.

Note: The last three figures are taken from Boresi, Schmidt and Sidebottom, "Advanced Mechanics of Materials", 5^{th} Ed, Wiley & Sons, 1993.